

Full Length Research Paper

Generalization and combination of Markowitz – Sharpe’s theories and new efficient frontier algorithm

Farhad Savabi^{1*}, Hamid shahrestani² and Bijan bidabad²

¹Islamic Azad University, Karaj Branch, Iran.

²Islamic Azad University, Science and Research Branch, Iran.

Accepted 23 December, 2011

Sharpe's efficient frontier or “capital market line” which is based on Markowitz's nonlinear efficient frontier consists of all different combinations of risky assets and is based on several assumptions, which displays the preference of this frontier to Markowitz's mean-variance efficient frontier (Busser, 1977; Merton, 1972). Although, this theory is effective for investing companies which holds unlimited amount of resources and huge amount of funds, it does not have any application for individuals who do not tend to borrow or lend and are willing to invest in limited number of shares. The most underlying purpose of this article is to present a mathematical model and to demonstrate the methods of investment in limited number of shares in a way that the unsystematic risk which the market does not reward is minimized.

Key words: Capital market line, capital asset pricing model (CAPM), efficient frontier, systematic risk, unsystematic risk, mean-variance analysis, linear programming.

INTRODUCTION

We have learnt that capital asset pricing model (CAPM) model is the development of Markowitz's model from a specific view and is the first theory in capital asset pricing models, which was presented by Sharp and Linter. In fact, the study of portfolio theory by Harry Markowitz, which demonstrates the relationship between risk and return was first published in 1952 and updated in 1959, provided the grounds for the emergence of the first theory in capital asset pricing model (CAPM) by William Sharpe.

Sharpe by choosing assumptions such as the followings succeeded in developing Markowitz's non-linear efficient frontier and changed it into linear form known as capital market line (CML) in the financial literature (Sharpe, 1963-1964; Markowitz, 1952, 1959; Lintner, 1965):

(i) Investors have homogenous expectations about asset returns,

(ii) There exists a free risk asset with R_f pure interest rate,

(iii) Investors may borrow or lend unlimited amounts at a constant rate.

(iv) All investors are price takers.

(v) Market portfolio not only exists, but also is measurable and lies on Markowitz's efficient frontier.

The main assumption in relation to CAPM model is the existence of market portfolio which was rejected by several financial researchers such as Roll (1997). In fact, the existence of this assumption enables us to derive the maximum amount for Sharpe ratio or the maximum reward for market systematic risk. According to a desired combination of market portfolios, one non-risky asset and the shape of indifference curves between risk and return, the investor is able to lie on one of the points lying on capital market line.

Needless to say that a risk averse investor tends to lend while a risk seeker investor tends to borrow. The purpose of this article is to combine Sharpe's and Markowitz's theories in order to, first, decrease the number of estimates in Markowitz's model; second, minimize the whole portfolio risk for individual investors

*Corresponding author. E-mail: farhad_savabi1@yahoo.com.

by presenting a mathematical model for a given level of expected return and for a typical portfolio.

According to Sharpe's theory, we observe the existence of systematic risk even in efficient combination risk, resulting from changes in economic activities. The existence of such risk justifies the demand for higher return by investors in order to enter into risky markets. Finally, by presenting a mathematical algorithm we will attempt to derive a new efficient frontier for these investors.

In addition, it is believed that one of the fundamental and basic criticisms of the CAPM model is assuming the selection of market portfolio by all investors while we believe one part of the investors only pay attention to one subset of market portfolio.

Under these conditions, the proposed model will absolutely remove the mentioned problem which is regarded as the contribution of the proposed model.

LITERATURE REVIEW

We have learnt that CAPM model is the development of Markowitz's model from a specific view and is the first theory in capital asset pricing models (Fama, 1996, 2004) which was presented by Sharpe and Lintner.

In fact, the study of portfolio theory by Harry Markowitz which demonstrates the relationship between risk and return was first published in 1952 and updated in 1959, provided the grounds for emergence of the first theory in capital asset pricing (CAPM) by William Sharpe.

Sharpe by choosing specific assumptions succeeded in developing Markowitz's nonlinear efficient frontier to linear efficient frontier, which is known as Capital Market Line (CML) in the financial literature. An important and remarkable starting point in the CAPM model is the assumption of the existence of market portfolio (M).

According to this assumption, market portfolio not only exists but also is computable and lies on Markowitz's efficient frontier. In fact, the existence of such assumption enables us to derive the maximum amount for Sharpe's ratio.

Sharpe's ratio or Sharpe's index in the financial literature is defined as follow:

$$S = \frac{E[R_p - R_f]}{\sqrt{\text{Var}[R_p - R_f]}} = \frac{E[R_p - R_f]}{\sigma_p} \quad (1)$$

If portfolio (p) lies in Markowitz's attainable region which is a region comprised of whole combinations of risky assets, an investor by assuming lending or borrowing with a pure interest rate, and the existence of free-risk asset with return equal to pure interest rate (R_f) will be able to create a linear efficient frontier through combining portfolio (p) and free risk asset THAT IS (R_f , P).

The slope of this line is equal to Sharpe's ratio and represents considered reward in exchange for acceptance of each unit of portfolio risk (p). Since market portfolio lies on the market efficient frontier, first the amount of Sharpe ratio is in its maximum level. Second, the whole risk of market portfolio is unsystematic risk (Evans and Archel, 1968). Third, market portfolio is a desired portfolio by all investors. Therefore, the Sharpe's ratio in this situation shows the maximum amount of considered reward on behalf of the market, in exchange for acceptance of each systematic risk unit.

As a result, an individual investor by choosing a desirable combination of market portfolio, one risk-free asset and according to the shape of his or her indifference curve in relation to the amount of risk and return will be able to lie on one of the points lying on the capital market line. Surely, as a risk averse investor he or she would tend to choose combinations lying between R_f M. While an investor with more aggressive spirit would tend to choose combinations lying along with Mk. These situations are displayed with indifference curves of u_1 and u_2 , respectively (Figure 1).

Another considerable point in this theory is the existence of linear relationship (Equation 2) as a necessary condition for maximizing relationship (Equation 3).

$$E(R_i) = R_f + \beta_{im}[E(Rm) - R_f] \quad (2)$$

This is known as security market line in the financial literature

$$MaxS = \frac{E[R_p - R_f]}{\sigma_p} = \frac{E[R_p - R_f]}{\sqrt{\text{Var}[R_p - R_f]}} \quad (3)$$

$$s.to : (1) \sum \omega_{ip} = 1$$

$$(2) \omega_{ip} \geq 0$$

The maximum amount in Sharpe ratio in Equation 3 will occur when portfolio (p) is equal to market portfolio (M).

In Equation 2, β_{im} represents the ratio of systematic risk of ith asset to market risk which is as follows:

$$\beta_{im} = \frac{\sigma_i}{\sigma_M} \rho_{iM}$$

ρ_{iM} represents the correlation between market return and return to ith asset, which is between zero and one:

$$0 \leq \rho_{iM} \leq 1$$

The main question is that if an individual investor is willing to invest in limited number of shares in portfolio M,

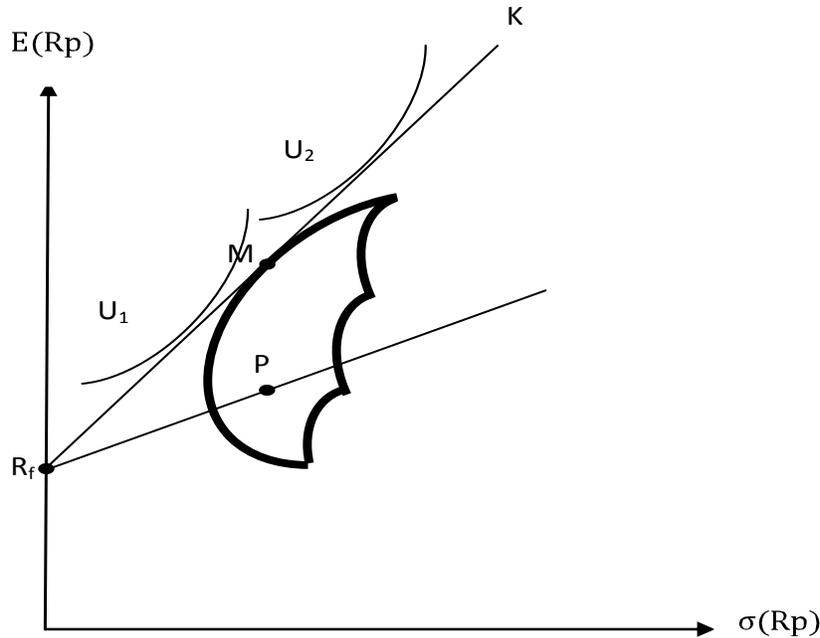


Figure 1. This Figure includes the relationship between risk and expected return for all risky assets.

how he or she will be able to minimize the amount of unsystematic risk, since according to CAPM model market only rewards systematic risk.

Another question which may arise is that whether it is possible to help this group of investors by presenting an appropriate algorithm, so that they can derive an efficient frontier. We try to address these questions by implementing Sharpe's and Markowitz's theories.

Proposed model

Based on Sharpe and Markowitz's theories, first, we intend to decrease the number of estimates in Markowitz's model, second, to minimize the proportion of unsystematic risk in a portfolio, and third, to present an algorithm to help individual investors in deriving the efficient frontier which we call "new efficient frontier" from now on.

Before starting the main discussion, we would like to remind you the CAPM's empirical model. We have:

$$R_{it} - R_{ft} = \beta_{iM}(R_{Mt} - R_{ft}) + \varepsilon_{it} \tag{4}$$

So that:

R_{it} represents ith risky return asset in the period of t.

R_{Mt} represents the market return in the period of t.

ε_{it} represents the error term related to ith risky asset in the period of t.

Regarding to ε 's, it is worth mentioning that all regression assumptions except the assumption of variance-error term equality govern the error term here.

Now according to CAPM empirical form and aforementioned assumptions for ε 's, it is possible to derive the following equation (Rao, 1989; Kevin, 2006).

$$\sigma_{ij} = \beta_{iM}\beta_{jM}\sigma^2_M + \sigma_{\varepsilon_i\varepsilon_j} \tag{5}$$

or:

$$\sigma_{ij} = \beta_{iM}\beta_{jM}\sigma^2_M, i \neq j \tag{6}$$

$$\sigma_i^2 = \beta_{iM}^2\sigma^2_M + \sigma_{\varepsilon_i}^2, i = j \tag{7}$$

So that:

σ_{ij} : Represent the covariance between rate of return on ith asset, and rate of return on jth asset.

σ^2_m : Represents market return rate variance.

$\sigma_{\varepsilon_i,\varepsilon_j}$: Represents the covariance between ith and jth error terms, respectively.

$\sigma^2_{\varepsilon_i}$: Represents ith error term variance.

Now based on obtained relationships in Equations 6 and 7, and according to CAPM model, it is possible to rewrite Markowitz's model as follows:

$$MaxE(Rp) = \sum_{i=1}^n \varpi_i E(R_i) \tag{8}$$

$$s.to : (1) \sum_{i=1}^n \varpi_i^2 \sigma_i^2 + \sum_{i=1}^n \sum_{j=1, j \neq i}^n \varpi_i \varpi_j \sigma_{ij} = \sigma_p^2$$

Table 1. This table displays the number of estimates in each of the models.

Situation	Number of estimates	Example n = 250
Markowitz's model	$n(n+3)/2$	31625
Sharpe's single factor model	$3n+2$	725
Proposed model	$2(n+1)$	502

$$(2) E(R_i) = R_f + \beta_{iM} (E(R_M) - R_f)$$

$$(3) \sigma_{ij} = \beta_{iM} \beta_{jM} \sigma_M^2 + \sigma_{\hat{\epsilon}_i \hat{\epsilon}_j}$$

$$(4) \sum_{i=1}^n \omega_i = 1$$

$$(5) \omega_i \geq 0$$

According to Equation 8 (Markowitz, 1987) the numbers of the estimates of the model are as follows:

- (i) n estimates related to β_{iM}
- (ii) 1 estimate related to σ_M^2
- (iii) 1 estimate related to $E(R_M)$
- (iv) n estimates related to $\sigma_{\hat{\epsilon}_i}$

Therefore, the whole number of estimates in the proposed model is:

$$n + 1 + 1 + n = 2n + 2 = 2(n + 1)$$

It can simply be seen that the whole number of estimates in the proposed model not only is less than Markowitz's model but also is less than Sharp's single factor model. Table 1 besides a numerical example can illustrate the subject.

Next, we intend to derive the new efficient frontier by presenting an algorithm and by using the proposed model.

NEW EFFICIENT FRONTIER ALGORITHM

Before presenting the algorithm, it is critical to remind you the following subjects which are repeatedly utilized in the financial literature. According to portfolio definition, beta for P is defined as follows:

$$\beta_p = \sum_{i=1}^n \omega_i \beta_i$$

According to the mentioned definition it can be written as:

$$\beta_p^2 = \left[\sum_{i=1}^n \omega_i \beta_i \right]^2 = (\omega_1 \beta_1 + \omega_2 \beta_2 + \dots + \omega_n \beta_n)^2 \quad (9)$$

Therefore, we have:

$$\beta_p^2 = \sum_{i=1}^n (\omega_i \beta_i)^2 + \sum_{i=1}^n \sum_{j=1, j \neq i}^n (\omega_i \beta_i)(\omega_j \beta_j)$$

And our basic assumption while presenting the mentioned algorithm is as follows:

If risk on ith asset is more than risk on jth asset ($\sigma_i^2 > \sigma_j^2$), then systematic and unsystematic risk on ith asset will be more than systematic and unsystematic risk on jth asset respectively. In other words, we have:

$$(\beta_{iM}^2 \sigma_M^2 > \beta_{jM}^2 \sigma_M^2 \ \& \ \sigma_{\hat{\epsilon}_i}^2 > \sigma_{\hat{\epsilon}_j}^2)$$

In order to derive the new efficient frontier again we consider the following relationships:

$$\sigma_p^2 = W^T Q W = \sum_{i=1}^n \omega_i^2 \sigma_i^2 + \sum_{i=1}^n \sum_{j=1, j \neq i}^n \omega_i \omega_j \sigma_{ij} \quad (10)$$

$$\sigma_{ij} = \beta_{iM} \beta_{jM} \sigma_M^2 + \sigma_{\hat{\epsilon}_i \hat{\epsilon}_j} \quad (11)$$

By replacing Equation 11 in Equation 10 and by using Equation 9, we will have:

$$\sigma_p^2 = \beta_p^2 \sigma_M^2 + \sum_{i=1}^n \omega_i^2 \sigma_{\hat{\epsilon}_i}^2 \quad (12)$$

The first sentence on the right hand side of Equation 12 represents the portion of whole risk on portfolio (p) which results from economic activity changes and the market is ready to reward it. While the second sentence on the right hand side of Equation 12 represents the unsystematic risk and is omissible through diversification and for this reason the market does not reward it.

We rearrange the equations as follows:

$$\text{Max} E(R_p) = \sum_{i=1}^n \omega_i [R_f + \beta_{iM} (E(R_M) - R_f)]$$

$$\sum_{i=1}^n [\varpi_i R_f + \varpi_i \beta_{iM} E(R_M) - \varpi_i \beta_{iM} R_f]$$

$$\sum_{i=1}^n [\varpi_i R_f + \sum_{i=1}^n \varpi_i \beta_{iM} E(R_M) - \sum_{i=1}^n \varpi_i \beta_{iM} R_f]$$

$$R_f + E(R_M) \sum_{i=1}^n \varpi_i \beta_{iM} - R_f \sum_{i=1}^n \varpi_i \beta_{iM}$$

$$R_f + \beta_p E(R_M) - \beta_p R_f$$

$$R_f + \beta_p [E(R_M) - R_f]$$

Therefore, we will rewrite Equation 13 as follows:

$$MaxE(R_p) = R_f + \beta_p [E(R_M) - R_f] \tag{13}$$

Now by considering Equations 12 and 13, Equation 8 can be rewritten as follows:

$$MaxE(R_p) = R_f + \beta_p [E(R_M) - R_f]$$

$$s.to : (1) \sigma_M^2 (\sum_{i=1}^n \varpi_i \beta_i)^2 + \sum_{i=1}^n \varpi_i^2 \sigma_{\epsilon i}^2 = \sigma_p^2$$

$$(2) \sum_{i=1}^n \varpi_i = 1 \tag{14}$$

$$(3) \varpi_i \geq 0$$

Since in Equation 15, the amount of R_i , and $E(R_M)$ are fixed, this statement will be maximized when β_p is in its maximum level.

Now by considering the conditions of Equation 14 relationship, it can simply be seen that if in $\sigma^2 p$ the amount of $\sum \varpi_i^2 \sigma_{\epsilon i}^2$ sentence is in its minimum level

then the amount of $\beta_p = \sum_{i=1}^n \varpi_i \beta_i$ will be in its maximum level.

According to the aforementioned interpretation, the only problem is:

$$Min Z = \sum \varpi_i^2 \sigma_{\epsilon i}^2$$

$$s.to : (1) \sigma_M^2 (\sum \varpi_i \sigma_i)^2 + \sum \varpi_i^2 \sigma_{\epsilon i}^2 = \sigma_p^2$$

$$(2) \sum \varpi_i = 1 \tag{15}$$

$$(3) \varpi_i \geq 0$$

It is clear that Equation 15 is a nonlinear programming model. By paying particular attention to Equation 15, it can be seen that the most important problem for us is to find the W vector so that we minimize the unsystematic portion of the risk. The amount of the systematic risk in W vector is of no importance to us.

Therefore, Equation 15 relationship changes to Equation 16:

$$Min Z = \sum \varpi_i^2 \sigma_{\epsilon i}^2$$

$$s.to : (1) \sum \varpi_i = 1 \tag{16}$$

$$(2) \varpi_i \geq 0$$

Now, we assume that:

$$A_i = \sigma_{\epsilon i}^2 < \sigma_{\epsilon (i+1)}^2 = A_{i+1} : i = 1, 2, \dots, (n-1)$$

Therefore, Equation 16 can be rewritten as follows:

$$Minz = A_1 \varpi_1^2 + A_2 \varpi_2^2 + \dots + A_n \varpi_n^2$$

$$s.to : (1) \sum \varpi_i = 1 \tag{17}$$

$$(2) \varpi_i \geq 0$$

By applying Lagrange coefficient method on Equation 17, the amount of weights will be as follows:

$$\varpi_i = \frac{A_1 A_2 \dots A_{i-1} A_{i+1} \dots A_n}{F} : i = 1, 2, \dots, n \tag{18}$$

As a result, F represents the summation on n sentences so that each sentence is one combination of $(n-1)$ element from a set of n element. Therefore, we have:

$$F = A_2 A_3 \dots A_n + A_1 A_3 \dots A_n + \dots + A_1 A_2 \dots A_{i-1} A_{i+1} \dots A_n + \dots + A_1 A_2 \dots A_{n-1} \tag{19}$$

According to the assumptions it can be easily seen that:

$$\varpi_i > \varpi_{i+1} : i = 1, 2, \dots, (n-1) \tag{20}$$

So, we will have W_1 vector follows:

$$W_1 = (\varpi_1, \varpi_2, \dots, \varpi_{n-1}, \varpi_n)$$

For this vector the amount of $\sigma^2 p$ will be $\sigma^2 p [w_1]$. In this level of $\sigma^2 p [w_1]$, the average weighted unsystematic risks of all risky assets that exit in the portfolio will be in minimum level with ϖ_i^2 weights.

So the first point, which is named, B_1 is obtained on the new efficient frontier as demonstrated in Figure 2.

$$E(R_p) = R_f + \frac{\sigma_p[W]}{\sigma_M} [E(R_M) - R_f] : \sigma_{\epsilon_j}^2 = 0, j = 1, \dots, n$$

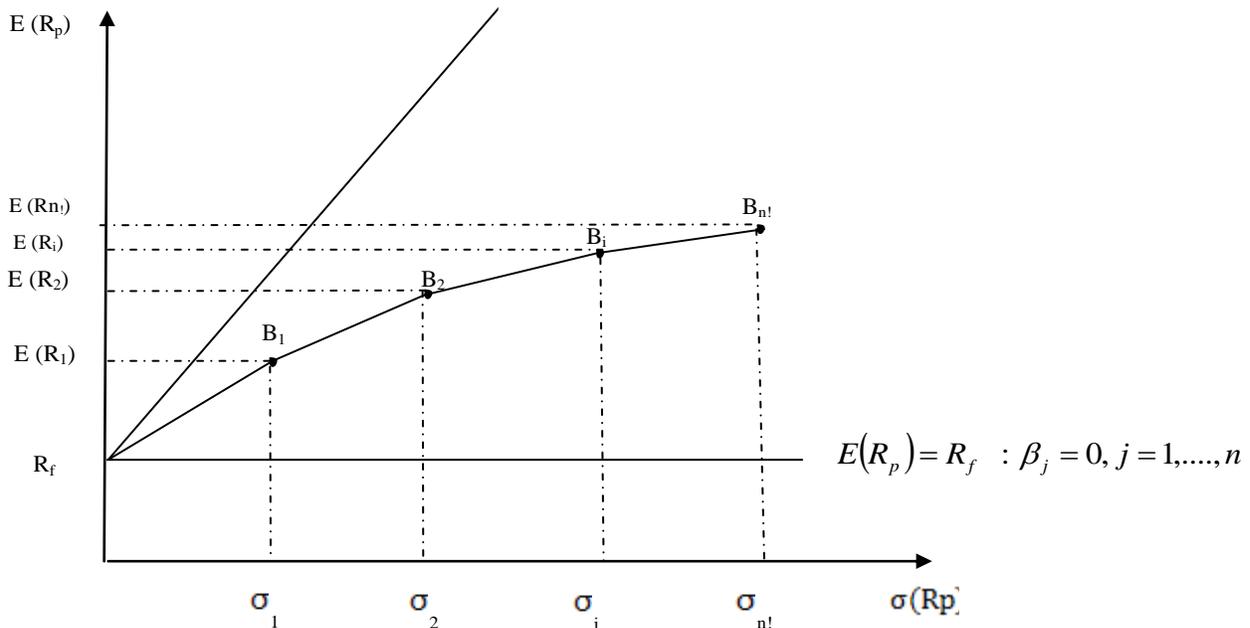


Figure 2. This figure shows the relationship between risk and expected return in the proposed model.

Driving other points lying on the new efficient frontier will be the next step. For this purpose, we shall change the arrangement of weights, so that new obtained vectors which are called $W_2, W_3 \dots W_{n!}$ respectively, will gradually increase the amount of Z in Figure 2. The reader may have found out that $W_{n!}$ vector will produce weights in the following requirements: It is unambiguously obvious that $W_{n!}$ vector maximizes the amount of z .

Therefore, according to Figure 2, $B_1, B_2, B_3 \dots B_{n!}$ points will lie on the new efficient frontier.

By considering Figure 2, an individual investor according to his or her utility function is able to choose one of the points lying on the new efficient frontier. For instance, if the given algorithm computer programming is available (Lewis, 1988), an individual investor, in a specific expected rate of return will not only be capable to find the percentages of investment in his or her given selected security, but also will be able to find the whole risk of his or her selected portfolio so that the portion of unsystematic risk will be minimized. Paying attention to Equation 15, the following results will be attainable:

- (i) If an investor selects a portfolio of stocks coincidentally, which their betas equal zero, then his or her efficient frontier according to Figure 2 will be a linear frontier with zero slope.
- (ii) If an investor selects a portfolio of stocks coincidentally so that $\sigma_{\epsilon_j}^2$ related to each stock equals zero then this

investor's efficient frontier according to Figure 2 will be a linear frontier with $\frac{E(R_M) - R_f}{\sigma_M}$ slope.

- (iii) Investing companies, due to high skills and huge amounts of funds always select a portfolio of risky assets which their risk is systematic. Therefore, they always lie on R_fK curve.

NUMERICAL EXAMPLE

Here, by considering hypothetical information related to three stocks and other information related to the market and non-risky asset returns and through similarizing pattern, we manage to derive the new efficient frontier mentioned in the previous part of the paper.

It is worth mentioning that derived expected returns in the proposed model against their whole related risks produced almost the same vector weights which were produced by GAMS software.

Presented model

Table 2 includes the information related to hypothetical securities. Assume the following information is available from the market:

Table 2. This table displays the amounts of systematic and unsystematic risk for three hypothetical securities

Security		
A	0.5	0.07
B	1	0.12
C	1.5	0.15

Table 3. This Table Shows the amounts of risk and expected return in the proposed model with three securities

W	σ_p	β_p	E(R _p)	B($\sigma_p, E(R_p)$)
W ₁ = ($\varpi_1 = 0.487807, \varpi_2 = 0.284552, \varpi_3 = 0.227642$)	$\sigma_{p1} = 0.3313$	$\beta_{p1} = 0.8699$	0.1617	B ₁ (0.3313, 0.1617)
W ₂ = ($\varpi_1 = 0.487806, \varpi_2 = 0.227642, \varpi_3 = 0.284552$)	$\sigma_{p2} = 0.3401$	$\beta_{p2} = 0.8983$	0.1657	B ₂ (0.3401, 0.1657)
W ₃ = ($\varpi_1 = 0.284552, \varpi_2 = 0.487806, \varpi_3 = 0.227642$)	$\sigma_{p3} = 0.3643$	$\beta_{p3} = 0.9715$	0.1760	B ₃ (0.3643, 0.1760)
W ₄ = ($\varpi_1 = 0.284552, \varpi_2 = 0.227642, \varpi_3 = 0.487806$)	$\sigma_{p4} = 0.3874$	$\beta_{p4} = 1.0284$	0.1839	B ₄ (0.3874, 0.1839)
W ₅ = ($\varpi_1 = 0.227642, \varpi_2 = 0.487806, \varpi_3 = 0.284552$)	$\sigma_{p5} = 0.4110$	$\beta_{p5} = 1.1016$	0.1942	B ₅ (0.4110, 0.1942)
W ₆ = ($\varpi_1 = 0.227642, \varpi_2 = 0.284552, \varpi_3 = 0.487806$)	$\sigma_{p6} = 0.8321$	$\beta_{p6} = 1.1300$	0.1982	B ₆ (0.8321, 0.1982)

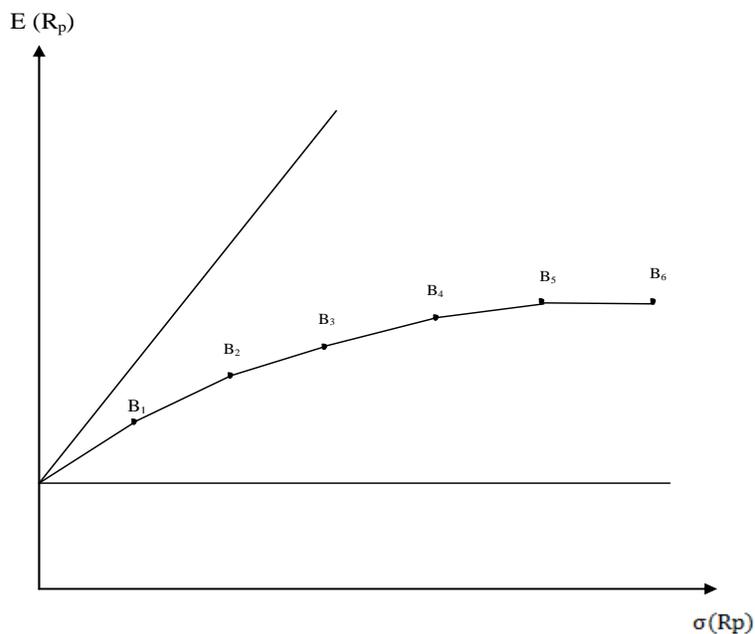


Figure 3. This Figure indicates risk and expected return in the proposed model with three securities.

$$E(R_M) = 0.18, \quad \sigma_M^2 = 0.1, \quad R_f = 0.04$$

$$(2) \varpi_1 + \varpi_2 + \varpi_3 = 1$$

After substituting the hypothetical amounts in the proposed model, we will have:

$$(3) \varpi_1, \varpi_2, \varpi_3 \geq 0$$

$$\begin{aligned} \text{Max } \beta_p &= 0.5\bar{\varpi}_1 + \bar{\varpi}_2 + 1.5\bar{\varpi}_3 \\ \text{s.t.} : (1) & 0.045\bar{\varpi}_1^2 + 0.22\bar{\varpi}_2^2 + 0.375\bar{\varpi}_3^2 \\ & + 0.1\bar{\varpi}_1\bar{\varpi}_2 + 0.15\bar{\varpi}_1\bar{\varpi}_3 + 0.3\bar{\varpi}_2\bar{\varpi}_3 = \sigma_p^2 \end{aligned}$$

By using Equation 14, weight amounts and other variables are as shown in Table 3. By drawing the points lying in the fifth column of Table 3, the new efficient frontier for investors will be as shown in Figure 3.

Conclusions

In this paper, we showed that the CAPM model, which is based on Markowitz's (1952, 1959) portfolio model, is a sound theoretical starting point. Though, many empirical studies by financial researchers like those of Fama and French refer to the weakness of the model, nevertheless, due to its extreme simplicity it has been widely used by financial managers and stock brokers.

According to this article, although the model can be beneficial for investing companies, it does not have any benefits for individual investors who do not intend to borrow and lend and are willing to invest their funds in a limited number of shares.

The most underlying purpose of this article was to present a mathematical model for this group of investors to invest their funds in a limited number of shares and to minimize their unsystematic risk, which the market does not reward. Another important purpose which was attained indirectly was to solve a nonlinear programming model by using linear means and methods.

Furthermore, we know one of the fundamental criticisms of the CAPM model is the assumption of selecting the market portfolio by all investors while we believe one part of investors only pay attention to one subset of market portfolio.

Under these conditions, the proposed model can remove the mentioned problem which can itself be considered the contribution of the proposed model.

Future areas for further research include implementing the presented model for all types of market indices, or for CAPM substitution models such as Fama and French three factor model, which is gradually considered to be prominent in today's financial literature.

REFERENCES

- Busser S (1977). "A Simplified Expression for the Efficient Frontier in Mean - variance Portfolio Analysis," *Manage. Sci.*, 23: 901-903.
- Evans J, Archer S (1968). "Diversification and the Reduction of Dispersion: An Empirical Analysis". *J. Finan.*, 23: 761-767.
- Fama EF, French KR (1996). "Multifactor Explanations of Asset Pricing Anomalies," *J. Finan.*, 51: 55-84.
- Fama EF, French KR (2004). "The Capital Asset Pricing Model: Theory and Evidence," *J. Finan.*, pp. 1-32.
- Keenan (2006). "Security Analysis and Portfolio Management," Prentice – hall of India private, New Dehli.
- Lewis A (1988). "A Simple Algorithm for the Portfolio Selection Problem," *J. Finan.*, 43: 71-82.
- Lintner J (1965). "The Valuation of Risk Assets and the Selection of Risky Investment in Stock Portfolios and Capital Budgets," *Rev. Econ. Stat.*, 47: 13-37.
- Markowitz H (1952). "Portfolio Selection". *J. Finan.*, 7: 77-91.
- Markowitz H (1959). "Portfolio Selection: Efficient Diversification of Investment". Cowles Foundation Monograph, No. 16. New York: John Wiley and sons, Inc.
- Markowitz H (1987). "Mean - variance Analysis in Portfolio Choice and Capital Markets," Cambridge, MA: Black well.
- Merton R (1972). "An Analytic Derivation of the Efficient Portfolio Frontier." *J. Financ. Quantit. Analys.*, 2: 1851-1872.
- Rao RKS (1989). "Fundamentals of Financial Management," Macmillan, New York.
- Roll R (1997). "A Critique of The Asset Pricing Theory's Tests PartI: On Past a Potential Testability of the Theory" *J. Financ. Econ.*, 4: 129-176.
- Sharpe WF (1964). "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk". *J. Financ.*, 19: 425-442.
- Sharpe WF (1963). "A Simplified Model for Portfolio Analysis" *Manage. Sci.*, 9: 227-243.